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Symmetries in neural networks: a linear group action approach

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Abstract. Neural networks storing Hadamard patterns have been completely classified with respect to permutation symmetry. The symmetry group of the Hadamard patterns is found to be isomorphic to $GL(n, F_2)$, and the symmetry groups of the networks are explicitly constructed for the most important classes. The volumes of different equivalence classes have been calculated.

Symmetry plays an important role in unification and classification of physical systems [1], e.g. crystals by their point group symmetry or the atomic nucleus by the isospin symmetry. In the first case, invariance under rotations and inversions in real space is considered and the knowledge of the invariance group is used in simplifying the calculation of the dynamical behaviour of the crystal, e.g. its phonon spectrum. In the second example, the invariance under the group $SU(2)$ leads to the multiplet representation of the nucleus structure. Permutation symmetry may also be important; a prominent example is the replica symmetry breaking in the spin glass theory, where a classification of the mean field theoretical solutions is achieved by consideration of their invariance under subgroups of the symmetric group [2].

Another object for which the study of symmetries proves to be very fruitful is the area of neural networks. Global properties of an individual network can be found from symmetry considerations of the invariance group of the specific pattern set stored by some learning rule [3, 4], e.g. the metastable states can be partitioned into orbits under the group action.

In the present paper a new approach to study symmetry properties of neural networks is proposed. It is general in the following aspects:

(i) Symmetry features of a network are found not one-by-one but according to the general results concerning the action of linear groups over finite fields. In particular it made it possible for us to find for Hopfield networks with Hadamard prototypes [5] all possible symmetries;

(ii) It is possible in this way to describe classes of networks with same symmetry features, which enables the study of the dynamical properties of networks belonging to one and the same class collectively;

(iii) The approach is also applicable to various learning rules as long as these do not take into account the numbering of patterns stored.

In the present paper we are going to demonstrate the advantages of this approach for the case of Hadamard pattern sets.

Hadamard patterns play an important role in optics [6–8] (for recovery of blurred images) as well as coding theory [9] (construction of error-correcting codes). The reason is that the orthogonal patterns of Hadamard type have a very rigid structure, which can be fully reconstructed even after a substantial damage. They can be defined recursively [5]; below we give a variant of such a construction.

For a vector $v = (v_0, v_1, \ldots, v_{m-1})$ from the space F_2^m, the vector space over the finite field F_2 of two elements 0, 1, we define its *complement* $\overline{v} = (1 - v_0, 1 - v_1, \ldots, 1 - v_{m-1})$. Thus $\overline{\overline{v}} = v$. For each n, let a set of vectors $H_n \subseteq F_2^{2^n}$ be defined inductively as follows. The set $H_0 = \{(0)\}$ and for $i \geqslant 0$, H_{i+1} consists of all vectors having the form (h, h) or $(h, \overline{h})$, where h is any vector from H_i. Thus H_{i+1} contains twice as many vectors as H_i, e.g.

$$
\begin{aligned}
H_1 &= \{(0,0), (0,1)\} \\
H_2 &= \{(0,0,0,0), (0,0,1,1), (0,1,0,1), (0,1,1,0)\} .
\end{aligned}
$$

We call H_n *the set of Hadamard vectors (patterns)* of length 2^n. Notice that the first vector component is always zero.

In general, H_n consists of 2^n vectors. These vectors can also be considered as elements of R^{2^n}, since F_2 can be embedded in any other field in a natural way. We shall specify the particular vector space when necessary.

We use the concept of (left) group action in its usual sense. The reader may find an introductory information on group action in the first chapter of the book [10]. Let S_d be the symmetric group of degree d, i.e. consisting of all permutations of the elements from the set $\{1, 2, \ldots, d\}$. For each subgroup G of S_{2^n-1} we introduce the action of G on $F_2^{2^n}$:

$$
G \times F_2^{2^n} \to F_2^{2^n} \qquad (\pi, h) \mapsto \pi h
$$

where for each $\pi \in G$ and $h \in F_2^{2^n}$, we set

$$
\pi h = (0, h_{\pi^{-1}(1)}, \ldots, h_{\pi^{-1}(2^n-1)}) .
$$

For example, for $n = 2$, the transposition $\pi = (1, 3) \in S_3$ and $h = (0, 0, 1, 1) \in H_2$ we have $\pi h = (0, 1, 1, 0)$.

The *symmetry group of* H_n is defined as the unique subgroup G_n of S_{2^n-1} consisting of all permutations which map Hadamard patterns to Hadamard patterns:

$$
G_n := \{\pi \in S_{2^n-1} \mid \pi H_n = H_n\} .
$$

Obviously, $G_1 = S_1$ and $G_2 = S_3$. In fact, $n = 1, 2$ are the only trivial cases with $G_n = S_{2^n-1}$. For example, $|G_3| = 168$ whereas $|S_{2^3-1}| = 5040$.

We have proved that for each $n \in N$, G_n is isomorphic to $GL(n, F_2)$, the group of all non-singular $n \times n$ matrices over F_2.

In addition, we have shown that the group isomorphism can be extended in a natural way such that the action of G_n on H_n is carried over one-to-one onto the action of $GL(n, F_2)$ on F_2^n. The latter action is by matrix multiplication to an F_2^n-vector. (We give all details of this isomorphism in our forthcoming paper [11].)

This fact, being the crucial point of our work, allows us to reduce the classification of Hadamard pattern sets with respect to the G_n-symmetry to the well understood study of certain aspects of the $GL(n, F_2)$-action.

For specific reasons we omit the trivial pattern $(0, 0, \ldots, 0)$:

$$\mathcal{H}_n := H_n \setminus \{\underbrace{(0, \ldots, 0)}_{2^n}\} \qquad \text{for each } n\,.$$

The set of all subsets of $\mathcal{H}_n$ with cardinality k is denoted as $\binom{\mathcal{H}_n}{k}$, i.e.

$$\binom{\mathcal{H}_n}{k} := \{S \mid S \subseteq \mathcal{H}_n\,,\ |S| = k\}\,.$$

Our main objects of interest are Hebbian neural networks in which an element $S \in \binom{\mathcal{H}_n}{k}$ has been stored. We call them Hadamard networks. For convenience we here switch from the $(0, 1)$ coding of the patterns to the $(1, -1)$ coding; the coupling matrix $J = ||J_{ij}||$ of such a network reads

$$J_{ij} = \sum_{\mu=1}^{k} h_i^\mu h_j^\mu$$

where $i, j = 1, \ldots, 2^n - 1$, $J_{0,0} = 1, J_{0,i} = J_{i,0} = \sum_{\mu=1}^{k} h_i^\mu$ and μ indexes some k Hadamard patterns $h^1, \ldots, h^k$. The action of the permutation group S_{2^n-1} on the set $\{1, \ldots, 2^n - 1\}$ naturally corresponds to an action on the networks; it is evident that the matrix of a network obtained from J by the action of a permutation π, $J' = \pi J$, reads

$$J'_{ij} = \sum_{\mu=1}^{k} h^\mu_{\pi^{-1}(i)} h^\mu_{\pi^{-1}(j)}$$

and is therefore equal to the matrix of a Hebbian network storing the correspondingly permuted patterns. For permutations π belonging to the group G_n the permuted patterns are also Hadamard and the resulting matrix is the matrix of some other Hadamard network, namely the one which stores patterns $\pi h^1, \pi h^2, \ldots, \pi h^k$. The action of G_n on H_n naturally extends to an action of G_n on $\binom{\mathcal{H}_n}{k}$ by $\pi\{h^1, h^2, \ldots, h^k\} = \{\pi h^1, \pi h^2, \ldots, \pi h^k\}$. Under this action $\binom{\mathcal{H}_n}{k}$ falls into orbits such that all networks storing pattern sets from the same orbit coincide up to a renumbering of neurons. Evidently, dynamic features of networks from one orbit coincide, e.g. the number of fixed points, mixed states, and retrieval properties. This leads to a natural classification of Hadamard networks by the orbits of the corresponding Hadamard pattern sets.

We next study the 'Hadamard pattern set classes', finding the number of classes $C^{n,k}$ and their orbit lengths $O_i^{n,k}$, as well as constructing their canonical representatives. Based on certain information about $GL(n, F_2)$ group conjugacy classes, the number of Hadamard pattern set classes may be computed by a variant of the Cauchy–Frobenius lemma ([10], pp 11, 79). Again, the details of this computation will be presented in [11]. In table 1 we give just the class numbers for small values of n (columns) and k (rows).

The following exact results on the class numbers have been proved [11]:
(i) for $0 < k < 2^n - 1$, $C^{n,2^n-1-k} = C^{n,k}$;
(ii) for $n > k$, $C^{n,k} = C^{k,k}$;
(iii) $C^{k,k} - C^{k-1,k} = 1$;
(iv) $C^{k-1,k} - C^{k-2,k} = k - 2$;
(v) $C^{k-2,k} - C^{k-3,k} = [(2k^3 + 21k^2 - 222k)/72] + 6$;

Table 1. Class numbers for small values of n (columns) and k (rows).

k	n = 1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	—	1	1	1	1	1	1	1	1	1
3	—	1	2	2	2	2	2	2	2	2
4	—	—	2	3	3	3	3	3	3	3
5	—	—	1	4	5	5	5	5	5	5
6	—	—	1	5	9	10	10	10	10	10
7	—	—	1	6	14	19	20	20	20	20
8	—	—	—	6	21	35	41	42	42	42
9	—	—	—	5	34	72	94	101	102	102
10	—	—	—	4	50	155	235	267	275	276
11	—	—	—	3	67	340	652	803	847	856
12	—	—	—	2	91	791	2076	2897	3163	3222
13	—	—	—	1	113	1907	7539	12637	14585	15025
14	—	—	—	1	129	4708	31500	68691	86625	90913
15	—	—	—	1	145	11780	149273	469936	683709	741393

where in (v) $[x]$ denotes the greatest integer less or equal x. These results are obtained by considering the Hadamard pattern sets as tuples of vectors from F_2^n. Then the differences $C^{k-i,k} - C^{k-i-1,k}$ turn out to be numbers of classes which, as vector tuples, have the same rank; for $i = 0, 1, 2$ they have been found by methods of linear algebra.

It can be seen from the table that the class numbers 'explode' even for modest values of n and k. However, most of the networks are contained in a very limited subset of classes. Namely, the following results have been obtained by direct calculation of the stabilizers (i.e. symmetry groups) of different classes.

(i) For k fixed and n going to infinity, a single class ('the winner') contains virtually all the networks [12], in fact the part of networks not belonging to that class decreases exponentially with n:

$$O_{\mathrm{W}}^{n,k}/O^{n,k} = \frac{(2^n-4)\cdots(2^n-2^{k-1})}{(2^n-3)\cdots(2^n-k)}$$

where $O_{\mathrm{W}}^{n,k}$ and $O^{n,k}$ denote the orbit length of the winner and the total number of pattern sets, respectively. The pattern sets of this class, considered as vector tuples, consist of independent vectors.

(ii) For $k = n$ (on the diagonal of the table) and n large the winner class contains about 29% of all networks: $O_{\mathrm{W}}^{n,n}/O^{n,n} \cong 0.288\,788$.

(iii) There exist exactly $k-2$ classes with rank $k-1$. Denoting their orbit lengths as $O_{k-1,d}^{n,k}$, $1 < d < k-1$, we have

$$O_{k-1,d}^{n,k} = \frac{|GL(n,F_2)|}{|GL(n-k+1,F_2)|}2^{-(k-1)(n-k+1)}\frac{1}{(d+1)!(k-d-1)!}.$$

The union of these orbits is

$$O_{k-1}^{n,k} = \frac{|GL(n,F_2)|}{|GL(n-k+1,F_2)|}2^{-(k-1)(n-k+1)}\left[\frac{2^k}{k!} - \frac{1}{(k-1)!} - \frac{1}{(k-2)!}\right].$$

(iv) For $k = n$, n large, the classes of rank $n-1$ contain about 58% of all networks: $O_{n-1}^{n,n}/O^{n,n} \cong 0.577\,58$.

(v) The classes of rank $k-2$ can be characterized by four parameters d_0, d_u, d_{I}, d_{II} satisfying $d_0 + d_u + d_{\mathrm{I}} + d_{\mathrm{II}} = k-2$, $0 \leqslant d_0$, $2 \leqslant d_{\mathrm{I}} \leqslant d_{\mathrm{II}}$, $d_u \leqslant (d_{\mathrm{I}}+1)/2$, with the corresponding orbits:

$$O^{n,k}_{k-2,d_0,d_u,d_{\mathrm{I}},d_{\mathrm{II}}} = \frac{|GL(n,F_2)|}{|GL(n-k+2,F_2)|} 2^{-(k-2)(n-k+2)} \frac{1}{d_0! d_u! (d_{\mathrm{I}}+1)! (d_{\mathrm{II}}+1)!} .$$

(vi) For $k = n$, n large, the classes of rank $n-2$ contain about 13% of all networks: $O^{n,n}_{n-2}/O^{n,n} = 0.128\,35$.

Taking (ii), (iv), and (vi) together, we see that on the diagonal all the classes not treated explicitly contain no more than just 0.5% of the networks.

Representatives of all the classes can also be obtained by methods similar to those of [10, ch 7]. However, there is no convenient algorithm to judge if two networks belong to one and the same class. Verification of this by direct application of the group action is impractical because the order of the group grows rapidly with n. However, there exists a simple method which answers the question in a vast majority of cases. Namely, we can analyse the distributions of the coupling weights of the two network matrices. The distribution of coupling weights is permutation invariant, so networks of the same class have equal distributions. The inverse holds for networks generated by pattern sets of rank k and $k-1$. For networks of the rank $k-2$ it has not been proven, but the classification can be accomplished by finding the invariants d_0, d_u, d_{I}, d_{II}.

An important quantity for the performance of networks with N neurons and k patterns is the number M^k_N, $N = 2^n$ of the fixed points of the network's dynamics. General results are known for random patterns [13] but also for Hadamard patterns [14]. An interesting point is that formulae could be derived for the cases $k = N - i$, $i = 1, 2, 3$ but not for $i \geqslant 4$. This can immediately be understood from property (i) of the class numbers showing that there are at least two classes for these cases instead of only one class. These two classes have different number of fixed points as the explicit result for $k = 12$, $N = 16$ [15] shows. On the other hand a general formula could be derived for the case $k = n$ with n odd [14]. Although for this case more than one class exist, one concludes that they have the same number of fixed points. We want to note that from the results for a network with $N = 16$ we see that Hadamard networks have in the mean a much larger number of fixed points than networks with the same number of random patterns. We attribute this to the larger symmetry group of Hadamard networks.

The ideas presented in this letter may be applied to other pattern sets constructed in a more general way:

(i) the neuron could instead of a two-state unit be a q-state unit, e.g. if one considers block codes for words over an alphabet of q symbols;

(ii) the iteration rule for constructing the patterns can be altered. In this way we can introduce other spaces than F_2; e.g. F_q if $q = p^r$ with p a prime number.

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